THERMOCAPILLARY CONVECTION IN A FLUID FILLING A HALF-SPACE

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The presence of a temperature gradient (occasioned by the dependence of the coefficient of surface tension on temperature) on a free fluid surface gives rise to tangential stresses on this surface. This in turn results in motion of the fluid – in so-called capillary convection [1].

We shall solve the problem of steady-state thermocapillary convection due to a point heat source of constant power situated on the free surface of a fluid filling a half-space,

Let a fluid fill the half-space $z > 0$. On the surface $z = 0$ of the fluid we have a point heat source of constant power Q. There is no gravitational field. (The latter condition is in a sense equivalent to the absence of thermal expansion of the fluid, since the Archimedeam force which occasions volume convection is absent both in the absence of a gravitational field and in a thermally undeformable fluid). We shall solve the problem in a spherical coordinate system whose origin coincides with the heat source and whose polar axis is directed along z. The angle ϑ is measured from the polar axis. The Eqs. and boundary conditions describing capillary convection can then be written as

$$(\mathbf{v}\nabla)\mathbf{v} = -\nabla(p/\rho) + \nu\Delta\mathbf{v}, \qquad \mathbf{v}\nabla T = \chi\Delta T, \qquad \operatorname{div}\mathbf{v} = 0 \tag{1}$$

The following conditions are satisfied on the free surface $\vartheta = \frac{1}{2}\pi$:

$$\eta\frac{1}{r}\frac{\partial v_r}{\partial\vartheta} = \frac{\partial\alpha}{\partial T}\frac{\partial T}{\partial r}, \quad v_\vartheta = 0, \quad \frac{\partial T}{\partial\vartheta} = 0 \quad \text{for} \quad r \neq 0 \tag{2}$$

The velocity and temperature must vanish at infinity.

The total thermal flux through the hemisphere with its center at the origin is a constant independent of the radius of the sphere,

$$\int_0^{2\pi} d\varphi \int_0^{\pi/2} \left[v_r T - \chi\frac{\partial T}{\partial r}\right] r^2 \sin\vartheta\, d\vartheta = \frac{Q}{\rho c_p} \tag{3}$$

Here η is the dynamic viscosity, χ the thermal diffusivity, α the coefficient of surface tension, ρ the density, and c_p the specific heat of the fluid at constant pressure.

Let the coefficient of surface tension α depend linearly on temperature. Then $\partial\alpha/\partial T =$ = const and the boundary value problem under consideration admits of the separation of variables

$$v_r = \frac{\Theta_1(\vartheta)}{r}, \quad v_\vartheta = \frac{\Theta_2(\vartheta)}{r}, \quad T = \frac{\Theta_3(\vartheta)}{r} \tag{4}$$

From the symmetry conditions we find that

$$v_\varphi = 0, \qquad \partial(\ldots)/\partial\varphi = 0 \tag{5}$$

Substituting (4) and (5) into boundary conditions (1) and (2) and eliminating the pressure, we obtain

$$(-\Theta_1^2 + \Theta_1'\Theta_2 - \Theta_2^2)' + 2\Theta_2\Theta_2' = \nu(\Theta_1'' + \Theta_1'\cot\vartheta - 2\Theta_1 - 2\Theta_2' - 2\Theta_2\cot\vartheta + \\ + 2\Theta_2'' + 2\Theta_2'\cot\vartheta + 4\Theta_1' - 2\Theta_2/\sin^2\vartheta)' \tag{6}$$

$$\Theta_1 + \Theta_2' + \Theta_2\cot\vartheta = 0 \tag{7}$$

$$-\Theta_1\Theta_3 + \Theta_2\Theta_3' = \chi\csc\vartheta(\Theta_3'\sin\vartheta)' \tag{8}$$

The conditions on the free surface $\vartheta = \frac{1}{2}\pi$ are:

$$\eta\Theta_1' = \Theta_3 \partial\alpha / \partial T, \quad \Theta_2 = 0, \quad \Theta_3' = 0 \qquad \text{for} \quad r \neq 0 \tag{9}$$

The condition of constant thermal flux is

$$2\pi \int_0^{\pi/2} (\Theta_1\Theta_3 + \chi\Theta_3) \sin\vartheta d\vartheta = \frac{Q}{\rho c_p} \tag{10}$$

From the symmetry conditions we find that

$$\Theta_1' = \partial p / \partial\vartheta = \Theta_3' = \Theta_2 = 0 \qquad \text{for} \quad \vartheta = 0 \tag{11}$$

The conditions of decrease at infinity are fulfilled automatically if we write the solution in form (4). Integrating Eqs. (6) and (8) once and eliminating Θ_1 by means of continuity Eq. (7), we obtain

$$-\Theta_2'^2 - 3\Theta_2'\Theta_2 \cot\vartheta + \Theta_2^2 - \Theta_2\Theta_2'' - \nu(-\Theta_2''' - 2\Theta_2'' \cot\vartheta + \Theta_2' \cot^2\vartheta - $$
$$-\Theta_2 \cot\vartheta / \sin^2\vartheta - 2\Theta_2 \cot\vartheta) + A\nu^2 = 0 \tag{12}$$

$$\Theta_2\Theta_3 = \chi\Theta_3' \tag{13}$$

Eq. (12) contains the integration constant $A\nu^2$. By virtue of conditions (11), the integration constant in Eq. (13) is equal to zero.

Let us integrate Eq. (12) once more. This yields

$$\Theta_2(\Theta_2 \sin\vartheta)' - \nu \sin\vartheta(\Theta_2'' + \Theta_2' \cot\vartheta - \Theta_2/\sin^2\vartheta + 2\Theta_2) - A\nu^2(1 - \cos\vartheta) = 0 \tag{14}$$

The integration constant was chosen here in accordance with condition (11). Eq. (14) admits of yet another integration,

$$\tfrac{1}{2}(\Theta_2 \sin\vartheta)^2 + \nu(-\Theta_2' \sin^2\vartheta + \Theta_2 \sin\vartheta \cos\vartheta) - \tfrac{1}{2}A\nu^2(1 - \cos\vartheta)^2 = 0 \tag{15}$$

Here the integration constant is chosen in such a way as to guarantee fulfillment of condition (11).

Eq. (15) can be transformed into

$$\frac{du}{dv} + u^2 = \frac{A}{4v^2} \qquad \left(u = \frac{\Theta_2}{2\nu \sin\vartheta}, \quad v = 1 + \cos\vartheta\right)$$

Let us rewrite this Eq. as

$$\exp\left(-\int u dv\right) \frac{d^2}{dv^2}\left(\exp\int u dv\right) = \frac{A}{4v^2} \tag{16}$$

Eq. (16) is an ordinary linear equation in the exponential. It is a special case of the equation derived by Slezkin [2] for jets associated with pulse sources. Solving it, we obtain

$$\exp\int u dv = \text{const}(v^{n_1} + Bv^{n_2}), \qquad n_{1,2} = \tfrac{1}{2}(1 \pm \sqrt{1 + A})$$

Returning to the function Θ_2, we find that

$$\Theta_2 = -\frac{A\nu \sin\vartheta}{2} \frac{(1 + \cos\vartheta)^{n_1 - 1} - (1 + \cos\vartheta)^{n_2 - 1}}{n_2(1 + \cos\vartheta)^{n_1} - n_1(1 + \cos\vartheta)^{n_2}} \tag{17}$$

The integration constant B was chosen in accordance with condition (9). Substituting (17) into (13), we find that

$$\Theta_3 = \frac{C}{[n_2(1 + \cos\vartheta)^{n_1} - n_1(1 + \cos\vartheta)^{n_2}]^{2P}}, \qquad P = \frac{\nu}{\chi} \tag{18}$$

Here P is the Prandtl number. The constant C can be determined from condition (9),

$$C = -\frac{A\nu\eta(1 + A)^P}{\partial\alpha / \partial T}$$

Finally, the constant A can be determined from the condition of a constant thermal flux (10). Substituting Θ_1 and Θ_3 into (10), we obtain

$$(1 + A)^P \int_1^2 \left[\frac{4P^2}{F^{2P+2}}\left(\frac{dF}{dv}\right)^2 v(2 - v) + \frac{1}{F^{2P}}\right] dv = -\frac{Q\, \partial\alpha / \partial T}{2\pi\nu\eta\kappa A}$$

$$F = n_1 v^{n_2} - n_2 v^{n_1} \tag{19}$$

Eq. (19) gives A in terms of the heat source power Q and of the fluid parameters, so that A for a given fluid depends only on Q.

From (19) we see that small Q correspond to small A. Expanding the integral in (19) in powers of A, we find that for small Q

$$A = -\frac{\partial \alpha / \partial T}{2\pi\nu\eta\varkappa} Q + \dots \qquad (20)$$

For arbitrary Q the integral in (19) cannot be investigated in general form. But for a given Prandtl number the function $A(Q)$ can be found by numerical integration in the required interval of A values.
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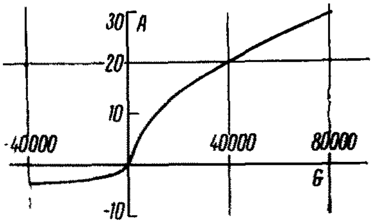

Fig. 1

Fig. 1 shows the curve of $A(Q)$ for water ($P = 7$). Clearly, A is a single-valued function of Q throughout the entire range of A values considered.

From (19) we see that the domain of existence of the integral in (19) is bounded with respect to A, i.e. for negative A values of sufficiently large absolute value the integral in (19) (and with it the whole of the solution obtained) is meaningless. This occurs for values of A for which $F(v)$ vanishes. The functions $\Theta_2(\vartheta)$ and $\Theta_3(\vartheta)$ then go to infinity and the integral in (19) loses its meaning. Let us find this terminal value of A.

We begin by showing that this value lies to the left of (-1). Let us set

$$\varepsilon = \sqrt{1 + A}$$

$$F(v) = (^1/_2\varepsilon - {}^1/_2)\, v^{1/2 + 1/2\varepsilon} + (^1/_2\varepsilon + {}^1/_2)\, v^{1/2 - 1/2\varepsilon} =$$
$$= \sqrt{v}\,[\varepsilon \cosh(^1/_2\varepsilon \ln v) - \sinh(^1/_2\varepsilon \ln v)], \qquad (1 \leqslant v \leqslant 2)$$

Clearly, $F(v)$ does not vanish for any v in the range $1 \leqslant v \leqslant 2$ provided that $A > -1$ (i.e. provided that ε is real and not equal to zero). At the point $\varepsilon = 0$ ($A = -1$) the resulting value is formally useless. This is because at the point $A = -1$ Eq. (16) has an adjoint solution, so that

$$\exp \int u\,dv = \mathrm{const}\,(v^{1/2} + B v^{1/2} \ln v) \qquad (21)$$

The functions $\Theta_2(\vartheta)$ and $\Theta_3(\vartheta)$ which can be found from (21) do not have singularities. Hence, a solution exists for $\varepsilon = 0$.

We may, however, disregard the adjoint solution and to obtain the solution for $\varepsilon = 0$ by taking the limit of the resulting solution (17) and (18) as $A \to -1$. The integral in (19) exists in this case. Thus, the point $\varepsilon = 0$ ($A = -1$) is not the terminal point for the solution obtained. Let us now consider $A < -1$, for which we have

$$F(v) = i\sqrt{v}\,[\,|\varepsilon| \cos(^1/_2 |\varepsilon| \ln v) - \sin(^1/_2 |\varepsilon| \ln v)] \qquad (1 \leqslant v \leqslant 2)$$

If $F(v) = 0$, then

$$\tan(^1/_2 |\varepsilon| \ln v) = |\varepsilon|$$

This Eq. has roots. The smallest value of $|\varepsilon|$ for which a root lies in the interval $1 \leqslant v \leqslant 2$ satisfies the condition

$$\tan(^1/_2 |\varepsilon_*| \ln 2) = |\varepsilon_*|$$

Hence, $|\varepsilon_*| = 3.73$, so that $A_* = -1 - |\varepsilon_*|^2 = -14.9$. Thus, the solution of the problem obtained above ceases to exist for $A = A_* = -14.9$.

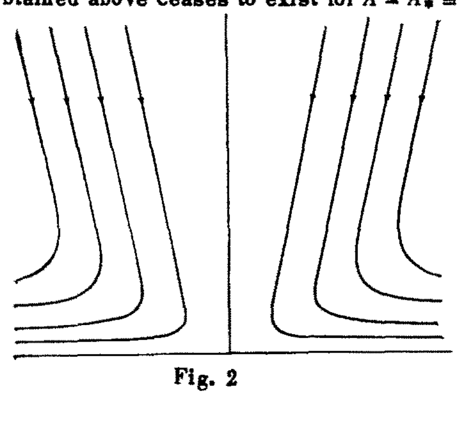

Fig. 2

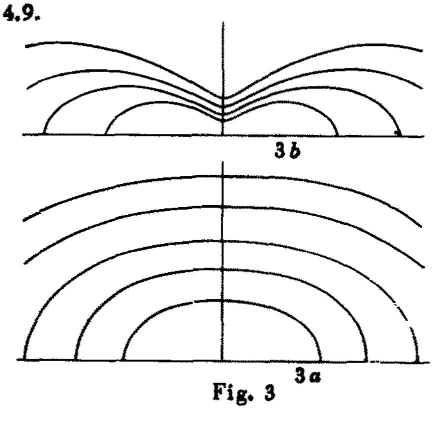

Fig. 3

Figs. 2 to 4 indicate the character of the motion and of the temperature field. The calculations were carried out for water. Fig. 2 shows the streamlines ($A = 2.24$) and Fig. 3 the isotherms for $A = 0.44$ (3*a*) and $A = 2.24$ (3*b*). Isotherm 3*a* corresponds to small Q. In this case heat transfer occurs largely through thermal diffusion. Figs. 2 and 3*b* afford a clear picture of the capillary convection mechanism: the capillary forces give rise to a characteristic spread of the fluid along the surface with "suck-in" toward the vertical. In spreading from the heat source along the surface the fluid takes the isotherms with it, so that the latter are extended along the surface, while the stream "sucked in" along the z-axis compresses the isotherms at the center. Finally, the shape of the isotherms indicates that a boundary layer is formed near the surface (a sufficient distance away from the heat source).

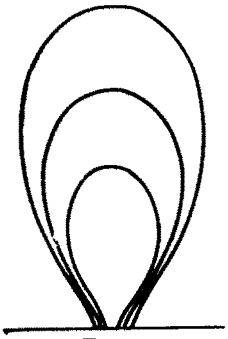

Fig. 4

For negative values of A the fluid flows along the surface toward the origin (toward the heat source), bunching the isotherms toward the vertical axis. It flows inward into the depths of the fluid along the vertical axis, taking the isotherms with it. This behavior of the isotherms is clearly apparent from Fig. 4, where they are shown for $A = -2.33$. The corresponding streamlines resemble those in Fig. 2, although the direction of motion is now reversed.

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